

On utility-based super-replication prices of contingent claims with unbounded payoffs

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Consider a financial market in which an agent trades with utility-induced restrictions on wealth. For a utility function which satisfies the condition of reasonable asymptotic elasticity at $-\infty$ we prove that the utility-based super-replication price of an unbounded (but sufficiently integrable) contingent claim is equal to the supremum of its discounted expectations under pricing measures with finite *loss-entropy*. For an agent whose utility function is unbounded from above, the set of pricing measures with finite loss-entropy can be slightly larger than the set of pricing measures with finite entropy. Indeed, the former set is the closure of the latter under a suitable weak topology.

Central to our proof is the representation of a cone C_U of utility-based super-replicable contingent claims as the polar cone to the set of finite loss-entropy pricing measures. The cone C_U is defined as the closure, under a relevant weak topology, of the cone of all (sufficiently integrable) contingent claims that can be dominated by a zero-financed terminal wealth.

We investigate also the natural dual of this result and show that the polar cone to C_U is generated by those separating measures with finite loss-entropy. The full two-sided polarity we achieve between measures and contingent claims yields an economic justification for the use of the cone C_U , and an open question.

1. Introduction

Consider a financial market where the discounted prices of d risky assets are modelled over a finite time interval $[0, T]$ by an \mathbb{R}^d -valued semimartingale $S = (S_t)_{0 \leq t \leq T}$, on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$ satisfying the usual conditions of right continuity and saturatedness. A portfolio on such a market can be represented by a pair (x, H) , consisting of an initial wealth $x \in \mathbb{R}$, and a predictable, S -integrable process H representing the holdings of the d risky assets. It is assumed that, at any time, all remaining wealth is invested in the numeraire. The discounted wealth process corresponding to the portfolio (x, H) is defined by $X_t^{(x, H)} := x + \int_0^t H_u dS_u$.

Two important theoretical concepts within the above framework for models of financial markets are those of “No Arbitrage” and completeness. An arbitrage opportunity is defined as a trading strategy H such that $X_T^{(0, H)} \geq 0$, \mathbb{P} -a.s. and such that $\mathbb{P}(X_T^{(0, H)} > 0) > 0$. A model is usually said to satisfy the condition of No Arbitrage if there does not

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exist an *admissible* trading strategy which is an arbitrage opportunity. The condition on H of admissibility is the requirement that the wealth process $X^{(0,H)}$ is uniformly bounded below by a constant; ruling out such processes is one way to disallow the use of doubling strategies.

In the celebrated papers [4, 5] the condition of No Arbitrage was weakened to that of No Free Lunch with Vanishing Risk (NFLVR), and it was shown that a model satisfies NFLVR if and only if the set, M_σ^e , of equivalent σ -martingale measures is non-empty.

A model satisfying NFLVR is then said to be complete if the set M_σ^e is a singleton (i.e. if $M_\sigma^e = \{\mathbb{Q}\}$). In a complete market model it is possible, with the use of an admissible trading strategy, to replicate and thereby uniquely price all contingent claims in $L^1(\mathbb{Q})$ with payoffs bounded from below (see [5, Theorem 5.16]). In contrast to this, if a market is incomplete there exist contingent claims with payoffs bounded from below which cannot be replicated by admissible trading strategies. For such contingent claims there exists an open interval of arbitrage-free prices, rather than a unique price.

Given a general contingent claim with payoff X , it is easy to see that an upper bound for the interval of arbitrage free prices is given by the super-replication price

$$\bar{\pi}(X) := \inf\{x \in \mathbb{R} : \text{there is an admissible } H \text{ such that } X \leq X_T^{(x,H)}\}. \quad (1.1)$$

As a special case of [5, Theorem 5.5] we know that for a contingent claim with payoff X bounded from below, the super-replication price is in fact the least upper bound for the interval of arbitrage free prices, in other words

$$\bar{\pi}(X) = \sup_{\mathbb{Q} \in M_\sigma^e} \mathbb{E}_{\mathbb{Q}}[X]. \quad (1.2)$$

However, for contingent claims with payoffs unbounded from below, admissible trading strategies are unsuitable for super-replication, and this dual representation of the super-replication price does not always hold. Indeed, [1, Example 8] constructs a market model and a contingent claim with payoff X unbounded from below such that

$$\bar{\pi}(X) > \sup_{\mathbb{Q} \in M_\sigma^e} \mathbb{E}_{\mathbb{Q}}[X]. \quad (1.3)$$

The intuitive reason for the breakdown of (1.2) is that the cone

$$K^{\text{adm}} := \left\{ X_T^{(0,H)} : H \text{ is admissible} \right\}.$$

is not closed with respect to a weak topology induced by the set of pricing measures.

It is useful at this point to extend slightly the definition of a super-replication price to allow terminal wealths from an arbitrary convex cone $K \subseteq L^0(\mathbb{P})$. Let

$$\bar{\pi}(X; K) := \inf\{x \in \mathbb{R} : X \leq x + Y \text{ for some } Y \in K\}.$$

Of course, $\bar{\pi}(X) = \bar{\pi}(X; K^{\text{adm}})$. Note that if K is a solid cone in a subspace F of $L^0(\mathbb{P})$ (i.e. $X \in F$ and $X \leq Y \in K$ implies $X \in K$) then

$$\bar{\pi}(X; K) := \inf\{x \in \mathbb{R} : X - x \in K\}.$$

If K is not solid then we may of course replace K by the smallest solid cone containing K without affecting the super-replication price.

We are now able to formulate the following natural question with (1.3) in mind: Given an arbitrary convex cone K of contingent claims, is it possible to find a minimal solid, closed convex cone $C \supseteq K$, and a suitable set M of pricing measures, such that

$$\bar{\pi}(X; C) = \sup_{\mathbb{Q} \in M} \mathbb{E}_{\mathbb{Q}}[X]? \quad (1.4)$$

A positive answer to this question was given in [1]. In this paper, preferences of an investor were incorporated in the construction of a weakly closed, enlarged cone C by means of the convex conjugate of the investor's utility function. The set M of measures consisted of those absolutely continuous separating measures with finite entropy. A dual representation of the form (1.4) was obtained for utility functions which are bounded from above. This result has since been extended in [8] to unbounded utility functions with Reasonable Asymptotic Elasticity at both $-\infty$ and $+\infty$, with subsequent alternative proofs given in [2], [3] and [7]. In this article we show that the use of finite loss entropy measures, as introduced in [8], allow us to go further by dropping the unnecessary condition that the utility function has Reasonable Asymptotic Elasticity at $+\infty$.

2. Assumptions on U

The following assumption holds throughout the paper:

ASSUMPTION 2.1 We assume that the investor has a critical wealth $a \in [-\infty, \infty)$ and a utility function $U : (a, \infty) \rightarrow \mathbb{R}$ which is increasing, strictly concave, continuously differentiable, and satisfies the Inada conditions

$$\lim_{x \downarrow a} U'(x) = \infty, \quad \lim_{x \uparrow \infty} U'(x) = 0. \quad (2.1)$$

Furthermore, if the domain of U is the whole real line (i.e. $a = -\infty$) then we assume that U has Reasonable Asymptotic Elasticity at $-\infty$, in the sense that

$$AE_{-\infty}(U) := \liminf_{x \rightarrow -\infty} \frac{xU'(x)}{U(x)} > 1.$$

This condition was introduced and discussed in detail in [10].

The convex conjugate V of the utility function U is defined for $y > 0$ by

$$V(y) = \sup_{x \in (a, \infty)} \{U(x) - xy\}.$$

It is well known (cf. [9, §26]) that under the conditions of Assumption 2.1, V is strictly convex, continuously differentiable and satisfies

$$V'(0) := \lim_{y \downarrow 0} V'(y) = -\infty \quad \text{and} \quad V'(\infty) := \lim_{y \uparrow \infty} V'(y) = -a. \quad (2.2)$$

The following Lemma, which is a simple consequence of [10, Prop. 4.1(iii)], provides an equivalent formulation of Reasonable Asymptotic Elasticity at $-\infty$.

LEMMA 2.2 *Let U be a utility function defined on the whole real line, and suppose that U has reasonable asymptotic elasticity at $-\infty$. Then there exists $b > 0$ such that V is positive and increasing on (b, ∞) and for any $\alpha > 1$ there exists $D > 0$ such that $V(\alpha y) \leq DV(y)$, for all $y \in (b, \infty)$.*

Proof. Since U has reasonable asymptotic elasticity at $-\infty$, a repeated application of [10, Prop. 4.1, (iii)] implies that there exist constants $y_0 > 0$, $\lambda > 1$ and $C > 0$ such that $V(\lambda^n y) \leq C^n V(y)$, for $y \geq y_0$ and $n \in \mathbb{N}$.

From (2.2) we see that $V'(\infty) = \infty$, so there exists a $b_0 > 0$ such that V is positive and increasing on (b_0, ∞) . Set $b := \max\{y_0, b_0\} > 0$. Given any $\alpha > 1$ there exists $n \in \mathbb{N}$ such that $\alpha \leq \lambda^n$. For $y \geq b$ we have $V(\alpha y) \leq V(\lambda^n y) \leq C^n V(y) \leq DV(y)$, where $D := C^n$. □

3. Finite Loss Entropy Measures

Relative to the cone K , we define also the convex set of *separating*, or *pricing* measures by

$$M_1 := \{\mathbb{Q} \ll \mathbb{P} : X \in L^1(\mathbb{Q}) \text{ and } \mathbb{E}_{\mathbb{Q}}[X] \leq 0 \text{ for all } X \in K\}. \quad (3.1)$$

In what follows, we refer frequently to the function $V^+ := \max\{V, 0\}$. Note however that in most places we can drop the $^+$, since V is convex, and its graph can be bounded from below by a straight line.

DEFINITION 3.1 (cf. [8]) *A measure $\mathbb{Q} \ll \mathbb{P}$ is said to have finite loss-entropy if there exists a constant $b > 0$ such that*

$$\mathbb{E}_{\mathbb{P}} \left[V^+ \left(\frac{d\mathbb{Q}}{d\mathbb{P}} \right) \mathbf{1}_{\{\frac{d\mathbb{Q}}{d\mathbb{P}} \geq b\}} \right] < \infty. \quad (3.2)$$

The set of pricing measures with finite loss-entropy is denoted by

$$\widehat{M}_V := \{\mathbb{Q} \in M_1 : \mathbb{Q} \text{ has finite loss entropy}\}. \quad (3.3)$$

REMARKS 3.2 (i) We use the $\widehat{}$ notation to distinguish the set of finite loss entropy measures from the set M_V of finite entropy measures used in related papers.

(ii) It is easy to see that if (3.2) holds for some constant $b > 0$ then it holds for all $b > 0$. In other words, if $\mathbb{Q} \in \widehat{M}_V$, then $\mathbb{E}_{\mathbb{P}} \left[V^+ \left(\frac{d\mathbb{Q}}{d\mathbb{P}} \right) \mathbf{1}_{\{\frac{d\mathbb{Q}}{d\mathbb{P}} \geq b\}} \right] < \infty$ for all $b > 0$.

(iii) If the domain of U is a half-line, we have $\widehat{M}_V = M_1$ (see [8, Remark 6.3]).

LEMMA 3.3 *The set \widehat{M}_V is convex.*

Proof. For the case where U is defined on a half real line (i.e. $a \in (-\infty, \infty)$) convexity is trivial, since $\widehat{M}_V = M_1$. Nevertheless, we give a universal proof. Since the function V^+ is convex and non-negative, given arbitrary constants $0 < y \leq z$, and $x, b > 0$ we have

$$\begin{aligned} V^+(y) \mathbf{1}_{[b, \infty)}(y) &\leq V^+(b) \mathbf{1}_{[b, \infty)}(y) + V^+(z) \mathbf{1}_{[b, \infty)}(y) \\ &\leq V^+(x) \mathbf{1}_{[b, \infty)}(x) + V^+(z) \mathbf{1}_{[b, \infty)}(z) + V^+(b). \end{aligned} \quad (3.4)$$

Take $\alpha \in [0, 1]$, $\mathbb{Q}_1, \mathbb{Q}_2 \in \widehat{M}_V$ and define $\mathbb{Q}_\alpha := \alpha \mathbb{Q}_1 + (1 - \alpha) \mathbb{Q}_2$. Let $b > 0$ be an arbitrary constant. Applying (3.4), we have

$$\begin{aligned} & \mathbb{E}_{\mathbb{P}} \left[V^+ \left(\frac{d\mathbb{Q}_\alpha}{d\mathbb{P}} \right) \mathbf{1}_{\{\frac{d\mathbb{Q}_\alpha}{d\mathbb{P}} \geq b\}} \right] \\ & \leq \mathbb{E}_{\mathbb{P}} \left[V^+ \left(\frac{d\mathbb{Q}_1}{d\mathbb{P}} \right) \mathbf{1}_{\{\frac{d\mathbb{Q}_1}{d\mathbb{P}} \geq b\}} \right] + \mathbb{E}_{\mathbb{P}} \left[V^+ \left(\frac{d\mathbb{Q}_2}{d\mathbb{P}} \right) \mathbf{1}_{\{\frac{d\mathbb{Q}_2}{d\mathbb{P}} \geq b\}} \right] + V^+(b) \\ & < \infty. \end{aligned} \quad \square$$

4. The Super-Replicable Contingent Claims

Let $L_U := \bigcap_{\mathbb{Q} \in \widehat{M}_V} L^1(\mathbb{Q})$ denote the vector space of all \widehat{M}_V -integrable contingent claims. Note that due to the definition of M_1 it follows that $K \subseteq L_U$. Consider the solid convex cone

$$K_U := \left\{ X \in L_U : X \leq \tilde{X} \text{ for some } \tilde{X} \in K \right\},$$

of all \widehat{M}_V -integrable contingent claims that can be dominated by a terminal wealth in K . We shall adopt throughout the common practise of identifying probability measures $\mathbb{Q} \ll \mathbb{P}$ with their Radon-Nikodym derivative $\frac{d\mathbb{Q}}{d\mathbb{P}} \in L_+^1(\mathbb{P})$. Following this convention we let L_V^+ denote the subspace of $L^1(\mathbb{P})$, formed by taking the linear span of the Radon-Nikodym derivatives of all finite loss entropy pricing measures. When endowed with the bilinear form,

$$\langle X, X^+ \rangle = \mathbb{E}_{\mathbb{P}} [X X^+],$$

it is easy to show that the pair (L_U, L_V^+) becomes a left dual system (cf. [8], [7]). We now define

$$C_U := \overline{K_U}^{\sigma(L_U, L_V^+)}.$$

The set C_U is the smallest solid, closed, convex cone in L_U containing K , and is useful for a duality theory. We use the subscript U to stress the weak dependence of this set upon U . In Section 6 we give a characterisation of C_U in terms of intersections of all $L^1(\mathbb{Q})$ closures of K_U . Following the terminology introduced in [8], we call $\pi(X; C_U)$ the *utility-based super-replication price* of X .

5. Main Results

Our main results are Theorems 5.2 and 5.3. Recall that if $\text{dom}(U) = \mathbb{R}$ then we assume only that U has Reasonable Asymptotic Elasticity at $-\infty$. Interestingly, we need neither the assumption that U is bounded, nor that U has Reasonable Asymptotic Elasticity at $+\infty$.

Given a non-empty set $A \subseteq L_U$ we let $\text{cone}(A)$ denote the smallest convex cone containing A and we define its *polar cone* $A^\triangleleft \subseteq L_V^+$ by

$$A^\triangleleft := \{X^+ \in L_V^+ : \langle X, X^+ \rangle \leq 0 \text{ for all } X \in A\}.$$

For a non-empty set $B \subseteq L_V^+$ we define in a similar way $\text{cone}(B)$, and the polar cone $B^\triangleleft \subseteq L_U$ by

$$B^\triangleleft := \{X \in L_U : \langle X, X^+ \rangle \leq 0 \text{ for all } X^+ \in B\}.$$

We recall now, without proof, some useful facts about polar cones (cf. [7]):

LEMMA 5.1 *Let $A \subseteq L_U$ be non-empty. Then $A^\triangleleft \subseteq L_V^+$ is a $\sigma(L_V^+, L_U)$ -closed convex cone. Moreover,*

$$(\text{cone}(A))^\triangleleft = \overline{A}^\triangleleft = A^\triangleleft \quad \text{and} \quad A^{\triangleleft\triangleleft} = \overline{\text{cone}(A)},$$

where closures are taken in the $\sigma(L_U, L_V^+)$ topology.

THEOREM 5.2 *Suppose that $\widehat{M}_V \neq \emptyset$. Then*

$$L_V^+ \cap \text{cone}(M_1) = \text{cone}(\widehat{M}_V) = K_U^\triangleleft = C_U^\triangleleft \quad (5.1)$$

and

$$C_U = (\widehat{M}_V)^\triangleleft. \quad (5.2)$$

Proof. To obtain (5.1), we first show that $L_V^+ \cap M_1 \subseteq \widehat{M}_V$. To this end, take any $\mathbb{Q} \in L_V^+ \cap M_1$. Since \mathbb{Q} is a probability measure, it follows from the definition of L_V^+ that $\mathbb{Q} = \alpha\mathbb{Q}_0 - (\alpha - 1)\mathbb{Q}_1$ for some $\mathbb{Q}_0, \mathbb{Q}_1 \in \widehat{M}_V$ and some $\alpha \geq 1$. We now show that $\mathbb{Q} \in \widehat{M}_V$:

In the case where U is defined on the whole real line, let $b > 0$ be the constant from the statement of Lemma 2.2. Due to Lemma 2.2, we have

$$\begin{aligned} \mathbb{E}_{\mathbb{P}} \left[V^+ \left(\frac{d\mathbb{Q}}{d\mathbb{P}} \right) \mathbf{1}_{\{\frac{d\mathbb{Q}}{d\mathbb{P}} \geq \alpha b\}} \right] &\leq \mathbb{E}_{\mathbb{P}} \left[V^+ \left(\alpha \frac{d\mathbb{Q}_0}{d\mathbb{P}} \right) \mathbf{1}_{\{\frac{d\mathbb{Q}_0}{d\mathbb{P}} \geq b\}} \right] \\ &\leq D \mathbb{E}_{\mathbb{P}} \left[V^+ \left(\frac{d\mathbb{Q}_0}{d\mathbb{P}} \right) \mathbf{1}_{\{\frac{d\mathbb{Q}_0}{d\mathbb{P}} \geq b\}} \right] < \infty, \end{aligned}$$

so $\mathbb{Q} \in \widehat{M}_V$. The case where U is defined on a half real line (i.e. $a \in (-\infty, \infty)$) is trivial due to Remark 3.2(iii).

Consequently,

$$L_V^+ \cap \text{cone}(M_1) \subseteq \text{cone}(L_V^+ \cap M_1) \subseteq \text{cone}(\widehat{M}_V).$$

Now take any $X \in K_U$ and any $\mathbb{Q} \in \widehat{M}_V$. There exists $\tilde{X} \in K \subseteq L^1(\mathbb{Q})$ such that $X \leq \tilde{X}$. Hence $\mathbb{E}_{\mathbb{Q}}[X] \leq \mathbb{E}_{\mathbb{Q}}[\tilde{X}] \leq 0$, and therefore, $\text{cone}(\widehat{M}_V) \subseteq K_U^\triangleleft$.

On the other hand, since $-L_+^\infty(\mathbb{P}) \cup K \subseteq K_U$,

$$\begin{aligned} K_U^\triangleleft &\subseteq (-L_+^\infty(\mathbb{P}))^\triangleleft \cap K^\triangleleft \\ &= \{X^+ \in L^1(\mathbb{P}) : \mathbb{E}_{\mathbb{P}}[XX^+] \geq 0 \text{ for all } X \in L_+^\infty(\mathbb{P})\} \cap K^\triangleleft \\ &= L_+^1(\mathbb{P}) \cap K^\triangleleft \\ &= \{X^+ \in L_+^1(\mathbb{P}) : X^+ \in L_V^+ \text{ and } \mathbb{E}_{\mathbb{P}}[XX^+] \leq 0 \text{ for all } X \in K\} \\ &= L_V^+ \cap \text{cone}(M_1). \end{aligned}$$

Moreover, applying Lemma 5.1

$$C_U^\triangleleft = (\overline{K_U}^{\sigma(L_U, L_V^+)})^\triangleleft = K_U^\triangleleft,$$

and (5.1) follows. A final application of Lemma 5.1 shows that

$$C_U = \overline{K_U}^{\sigma(L_U, L_V^+)} = K_U^{\triangleleft\triangleleft} = (\text{cone}(\widehat{M}_V))^\triangleleft = (\widehat{M}_V)^\triangleleft. \quad \square$$

THEOREM 5.3 Suppose that $\widehat{M}_V \neq \emptyset$. Then for any $X \in L_U$,

$$\bar{\pi}(X; C_U) = \sup_{\mathbb{Q} \in \widehat{M}_V} \mathbb{E}_{\mathbb{Q}}[X].$$

Proof. Due to Theorem 5.2, $C_U = (\widehat{M}_V)^\triangleleft$. Since C_U is solid in L_U we have

$$\begin{aligned} \bar{\pi}(X; C_U) &= \inf\{x \in \mathbb{R} : X - x \in C_U\} \\ &= \inf\{x \in \mathbb{R} : \mathbb{E}_{\mathbb{Q}}[X - x] \leq 0 \text{ for all } \mathbb{Q} \in \widehat{M}_V\} \\ &= \inf\{x \in \mathbb{R} : \mathbb{E}_{\mathbb{Q}}[X] \leq x \text{ for all } \mathbb{Q} \in \widehat{M}_V\} \\ &= \sup_{\mathbb{Q} \in \widehat{M}_V} \mathbb{E}_{\mathbb{Q}}[X]. \end{aligned} \quad \square$$

REMARK 5.4 Given that C_U is a $\sigma(L_U, L_V^+)$ -closed cone, it is specified by its polar set, $\text{cone}(\widehat{M}_V)$. This set depends only on the shape of $V(y)$ for arbitrarily large y , which in turn depends only on the values of $U(x)$ for arbitrarily large negative x . Consequently, the cone C_U of allowable terminal wealths only depends on the preferences of the investor to asymptotically large losses. This interesting observation also suggests an open problem: Can the set C_U be parametrised by a real number which is defined in terms of the asymptotic behaviour of U at $-\infty$?

6. A Representation of C_U

Note that the set K_U of section 4 can be rewritten as

$$K_U = \bigcap_{\mathbb{Q} \in \widehat{M}_V} (K - L_+^1(\mathbb{Q})). \quad (6.1)$$

The next theorem gives two useful alternative representations of the weak closed cone C_U .

THEOREM 6.1

$$C_U \stackrel{(i)}{=} \bigcap_{\mathbb{Q} \in \widehat{M}_V} \overline{K_U}^{L^1(\mathbb{Q})} \stackrel{(ii)}{=} \bigcap_{\mathbb{Q} \in \widehat{M}_V} \overline{K - L_+^1(\mathbb{Q})}^{L^1(\mathbb{Q})}.$$

Proof. (i) To show one inclusion, let $X \in \bigcap_{\mathbb{Q} \in \widehat{M}_V} \overline{K_U}^{L^1(\mathbb{Q})}$. Then for each $\mathbb{Q} \in \widehat{M}_V$ there exists a sequence $\{X_n^{\mathbb{Q}}\} \subseteq K_U$ such that $X_n^{\mathbb{Q}} \xrightarrow{L^1(\mathbb{Q})} X$ as $n \rightarrow \infty$. Since $K_U \subseteq (\widehat{M}_V)^\triangleleft$ it follows that $\mathbb{E}_{\mathbb{Q}}[X] = \lim_{n \rightarrow \infty} \mathbb{E}_{\mathbb{Q}}[X_n^{\mathbb{Q}}] \leq 0$ for each $\mathbb{Q} \in \widehat{M}_V$. Consequently, $X \in (\widehat{M}_V)^\triangleleft = C_U$.

For the other inclusion, we proceed along the lines of the proof of the Kreps-Yan Theorem (cf. [6, Theorem 3.5.8]) and consider an arbitrary $Z \in L_U$ such that $Z \notin \overline{K_U}^{L^1(\mathbb{Q}^*)}$ for some $\mathbb{Q}^* \in \widehat{M}_V$. By the Hahn Banach Hyperplane Separation Theorem there exists a continuous linear functional on $L^1(\mathbb{Q}^*)$ that separates Z from the closed cone $\overline{K_U}^{L^1(\mathbb{Q}^*)}$. In other words there exists a $\Lambda \in L^\infty(\mathbb{Q}^*)$ such that

$$\mathbb{E}_{\mathbb{Q}^*}[\Lambda X] \leq 0 < \mathbb{E}_{\mathbb{Q}^*}[\Lambda Z] \quad (6.2)$$

for all $X \in K_U$. By considering $X = -\mathbf{1}_{\{\Lambda < 0\}} \in -L_+^\infty(\mathbb{P}) \subseteq K_U$, we see that $\Lambda \geq 0$ \mathbb{Q}^* -a.s. and $\mathbb{E}_{\mathbb{Q}^*}[\Lambda] > 0$. Thus, if we set $\Lambda^* = \Lambda / \mathbb{E}_{\mathbb{Q}^*}[\Lambda]$ then $\mathbb{Q}_0(A) := \mathbb{E}_{\mathbb{Q}^*}[\Lambda^* \mathbf{1}_A]$ defines a probability measure on (Ω, \mathcal{F}) , and (6.2) implies that $\mathbb{Q}_0 \in M_1$ and $\mathbb{E}_{\mathbb{Q}_0}[Z] > 0$. To finish the proof of the first inequality it suffices to prove that \mathbb{Q}_0 has finite loss entropy, as then it follows from (6.2) that $\mathbb{Q}_0 \in \widehat{M}_V$ and $Z \notin (\widehat{M}_V)^\triangleleft = C_U$.

In the case where U is defined on the whole real line, let $b > 0$ be the constant from the statement of Lemma 2.2. Due Lemma 2.2 it follows that

$$\begin{aligned} \mathbb{E}_{\mathbb{P}} \left[V^+ \left(\frac{d\mathbb{Q}_0}{d\mathbb{P}} \right) \mathbf{1}_{\left\{ \frac{d\mathbb{Q}_0}{d\mathbb{P}} \geq b \|\Lambda^*\|_{L^\infty(\mathbb{Q}^*)} \right\}} \right] &= \mathbb{E}_{\mathbb{P}} \left[V^+ \left(\Lambda^* \frac{d\mathbb{Q}^*}{d\mathbb{P}} \right) \mathbf{1}_{\left\{ \Lambda^* \frac{d\mathbb{Q}^*}{d\mathbb{P}} \geq b \|\Lambda^*\|_{L^\infty(\mathbb{Q}^*)} \right\}} \right] \\ &\leq \mathbb{E}_{\mathbb{P}} \left[V^+ \left(\|\Lambda^*\|_{L^\infty(\mathbb{Q}^*)} \frac{d\mathbb{Q}^*}{d\mathbb{P}} \right) \mathbf{1}_{\left\{ \frac{d\mathbb{Q}^*}{d\mathbb{P}} \geq b \right\}} \right] \\ &\leq D \mathbb{E}_{\mathbb{P}} \left[V^+ \left(\frac{d\mathbb{Q}^*}{d\mathbb{P}} \right) \mathbf{1}_{\left\{ \frac{d\mathbb{Q}^*}{d\mathbb{P}} \geq b \right\}} \right] \\ &< \infty. \end{aligned}$$

The case where U is defined on a half real line (i.e. $a \in (0, \infty)$) is trivial due to Remark 3.2(iii).

(ii) To prove the second equality it suffices to show that

$$\overline{K_U}^{L^1(\mathbb{Q})} = \overline{K - L_+^1(\mathbb{Q})}^{L^1(\mathbb{Q})},$$

for an arbitrary $\mathbb{Q} \in \widehat{M}_V$. Indeed, from (6.1) we have $K_U \subseteq K - L_+^1(\mathbb{Q}) \subseteq L^1(\mathbb{Q})$, so $\overline{K_U}^{L^1(\mathbb{Q})} \subseteq \overline{K - L_+^1(\mathbb{Q})}^{L^1(\mathbb{Q})}$. Moreover, since $K \cup (-L_+^\infty(\mathbb{Q})) \subseteq K_U$, we have $K - L_+^\infty(\mathbb{Q}) \subseteq K_U$. Since $L^\infty(\mathbb{Q})$ is dense in $L^1(\mathbb{Q})$, it follows that

$$\overline{K - L_+^1(\mathbb{Q})}^{L^1(\mathbb{Q})} = \overline{K - L_+^\infty(\mathbb{Q})}^{L^1(\mathbb{Q})} \subseteq \overline{K - L_+^\infty(\mathbb{Q})}^{L^1(\mathbb{Q})} \subseteq \overline{K_U}^{L^1(\mathbb{Q})}. \quad \square$$

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